23p.

X63-11421

NASA TT F-8213
Code 2d

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ON PROBABILITIES FOR EXTREME VALUES OF SUMS OF RANDOM VARIABLES DEFINED ON A HOMOGENEOUS MARKOV CHAIN WITH A FINITE NUMBER OF STATES

by I. S. Volkov

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON April 1962

ON PROBABILITIES FOR EXTREME VALUES OF SUMS OF RANDOM VARIABLES DEFINED ON A HOMOGENEOUS MARKOV CHAIN WITH A FINITE NUMBER OF STATES

[Following is a translation of an article by I. S. Volkov in Teoriya Veroyatnostey i Yeye Primeneniya (Probability Theory and Its Applications), Vol V, No. 3, Moscow, 1960, pages 338 - 352.]

1. Statement of the Problem

Let $E_1 E_2, \dots, E_m$ be the set of possible states of a simple homogeneous Markov chain; $e(k)$ the state of the chain at the instant of time $k(k = 0, 1, 2, \dots)$; $P = \quad p_{ij}$ the transition probability matrix. We shall give on the states $e(k)$ the unique function $f(e(k))$:

$$f(e(k)) = s_i, \text{ if } e(k) = E_i, \; k \quad 0,$$

where s_i are non-negative integers which do not depend on k and use $P_{pj}(n,s)$ to denote the probability

$$P \quad {}_{k=0}^{n} \; f(e(k)) = s, e(n) = E_j \quad e(0) = E_q \quad .$$

We shall hold to the following definitions ([1], [2], [5]): the sequence of states

$$(E_{i_1} E_{i_2} \dots E_{i}) \qquad (1.1)$$

is called the chain if $p_{i_j i_{j+1}} \quad 0 \; (1 \quad j \quad - 1)$;

$f(E_{i_1}) + f(E_{i_2}) + \dots + f(E_{i})$ is the weight of the chain (1.1);

the length of the chain (1.1); the chain (1.1) is called a cycle if $p_{i \; i_1} \quad 0$;

$\frac{1}{} [f(E_{i_1}) + f(E_{i_2}) + \dots + f(E_{i})]$ is the specific weight of the cycle (1.1); and are the smallest and greatest specific weights respectively of those cycles (1.1) for which the following conditions are satisfied:

a) E_{i_1} is attainable from E_q,

b) E_j is attainable from E_{i_1},

c) All states $E_{i_1}, E_{i_2}, \dots, E_{i}$ are different.

Henceforth we shall consider only those cycles which satisfy requirements a) and b); we shall call cycles with specific weight minimal and those with specific weight maximal.

For the sequence of random variables $\left\{\sum_{k=0}^{n} f(e(k))\right\}$ $(n = 0, 1, \ldots)$ the values $\delta n + \gamma$, $\Delta n + \gamma$ (γ = const) are extreme in the sense that, on one hand, for all sufficiently large the following relationship is satisfied identically on n:

$$P_{qj}(n, \delta n - \Gamma) = 0; \quad P_{qj}(n, \Delta n + \Gamma) = 0$$

and at the same time, as implied by the definitions of δ and Δ, there exist constants Γ_1, Γ_2 such that the following inequality holds:

$$\limsup_{n \to \infty} P_{qj}(n, \delta n + \Gamma_1) > 0; \quad \limsup_{n \to \infty} P_{qj}(n, \Delta n + \Gamma_2) > 0.$$

A study will be made in this article on the asymptotic behavior of the probabilities $P_{qj}(n, \delta n + \gamma)$, $P_{qj}(n, \Delta n + \gamma)$ (where γ is an arbitrary but fixed number) as $n \to \infty$. It will be established that the quantity $P_{qj}(n, \delta n + \gamma)$ can be represented in the form of a sum of a finite number of components $P_\beta^{(n)}(\gamma_\beta)$ which do not depend on n and which possess the following properties;

a) If $n \not\equiv \gamma_\beta \pmod{J_\beta(\gamma_\beta)}$, then $P_\beta^{(n)}(\gamma_\beta) = 0$,
b) If $n_i \to \infty$ when $i \to \infty$ by the law

$n_i \equiv \nu + \gamma_\beta \pmod{I_\beta(\gamma_\beta)}$, $\nu \equiv 0 \pmod{J_\beta(\gamma)}$, then the ratio

$\dfrac{P_\beta^{(n_i)}(\gamma_\beta)}{n_i^{R_\beta(\gamma_\beta)-1}\Lambda_\beta^{n_i}(\gamma_\beta)}$ tends to a finite limit $(\ )(\ \ )$ which is larger than zero.

The complete definition of all the quantities introduced here requires a number of preliminary considerations. We note here only that the basis for separating the components $p_\beta^{(n)}(\gamma_\beta)$ is the possibility of decomposing the set of all chains of length n + 1 and weight $\delta n + \gamma$ into nonintersecting pencils of trajectories and observing the rule: the trajectories of one pencil should be obtained from a fixed chain by adding minimal cycles at certain places.

It will be proved in this article that an exact expression for $P_{qj}(n, \delta n + \gamma)$ has the form of a sum of a finite number of terms $a_k n^{R_k} \lambda_k^n$ which do not depend on n; the values of the constants a_k, R_k, and λ_k are found by the method of generating functions.

Analogous results can be obtained for $P_{qj}(n, \quad .n + \quad)$ from the statements formulated above if we make use of the formula

$$P_{qj}(n, \quad n + \quad) = P \quad {}_{k=0}^{n} \quad {}_{i=1}^{m} s_i - f(e(k)) \quad =$$

$$= \quad {}_{i=1}^{m} s_i - \quad .n + \quad {}_{i=1}^{m} s_i - \quad , e(n) = E_j \quad e(0) = E_q \quad ,$$

having noted that the cycles of a chain which are maximal for given values of s_i turn out to be minimal on passing from s_i to

$$s_i = \quad {}_{k=1}^{m} s_k - s_i.$$

The author is grateful to A. N. Kolmogorov and B. A. Sevast'yanov for valuable assistance.

2. The Construction of Chains of Length $n + 1$ and Weight $\quad n +$

1°. We shall give a mapping of the set of all chains of finite length into itself in the following manner: in an arbitrarily chosen chain (1.1) we note all repetitions of the initial state (in the given case E_{i_1}); let them be observed at $k_1, k_2, \ldots, k_t$ places; from among the cycles

$$(E_{i_1}E_{i_2}\ldots E_{i_s}); \quad (s = k_1 - 1, k_2 - 1, \ldots, k_t - 1) \qquad (2.1)$$

we single out the longest minimal cycle -- say $(E_i \ldots E_{i \quad -1})$ and remove it from the chain (1.1); we then obtain another chain, namely $(E_{i_1}, E_i \quad , E_{i \quad +1}, \ldots, E_i \quad)$. By definition,

$$(E_{i_1}E_{i_2}\ldots E_i \quad) = (E_{i_1}E_i \quad E_{i \quad + 1}\ldots E_i \quad).$$

If there are no minimal cycles (in particular if there are in general no repetitions of the state E_{i_1}) among the cycles (2.1), then

$$(E_{i_1}\ldots E_i \quad) = (E_{i_1}E_{i_2}\ldots E_i \quad).$$

2°. Let us take one of the chains of length $n + 1$ and weight of $n + \quad$:

$$(E_{i_0}E_{i_1}\ldots E_{i_n}). \qquad (2.2)$$

Let $\Theta(E_{i_0}E_{i_{\sigma_1}}\dots E_{i_n}) = (E_{i_0}E_{i_{\sigma_1}}E_{i_{\,1+1}}\dots E_{i_n})$,

$\Theta(E_{i_{\sigma_1}}E_{i_{\sigma_1+1}}\dots E_{i_n}) = (E_{i_{\sigma_1}}E_{i_{\sigma_2}}E_{i_{\sigma_2+1}}\dots E_{i_n})$,

$\Theta(E_{i_{\sigma_{l-1}}}E_{i_{\sigma_{l+1}+1}}\dots E_{i_n}) = (E_{i_{\sigma_{l-1}}}E_{i_n})$.

The sequence of states E_{i_0}, $E_{i_{\sigma_1}}$, $E_{i_{\sigma_2}}$, ..., $E_{i_{\sigma_{l-1}}}$, E_{i_n} forms a chain; we shall call it the reduced chain corresponding to the given chain (2.2), and more generally, to the given value of n (with fixed).

Each reduced chain $(E_{\sigma_0}E_{\sigma_1}\dots E_{\sigma_l})$ possesses the following properties:

a) If $E_{\sigma_i} = E_{\sigma_k}$ $(0 \leq i \;\; k \leq l)$, then the specific weight of the cycle $(E_{\sigma_i}E_{\sigma_{i+1}}\dots E_{\sigma_{k-1}})$ is greater than δ;

b) $\sum_{i=0}^{l} f(E_{\sigma_i}) = \delta l + \gamma$.

Conversely, any of the chains which satisfy these two requirements is a reduced chain; it still remains, however, to discover which one of them corresponds to which values of n.

3°. We shall use $\mathcal{M}_{qj}(\gamma)$ to denote the set of all chains beginning at E_q, ending at E_j, and possessing properties a) and b) formulated in Point 2 of Section 2. We shall convince ourselves that the set $\mathcal{M}_{qj}(\gamma)$. On the strength of property b), this would mean that

$$\lim_{i \to \infty} \frac{S_i}{L_i} = \delta \tag{2.3}$$

where S_i is the weight of the separated chain with number i. However, inasmuch as any chain of length m + 1 contains at least one cycle, then, in view of property a), (2.3) contradicts the minimality of the specific weight δ.

We note that the set $\mathcal{M}_{qj}(\gamma)$ can be empty. In this case $P_{qj}(n, \delta n + \gamma) = 0$ for all natural values of n.

Henceforth, we shall consider that the set $\mathcal{M}_{qj}(\gamma)$ consists of m $(m > 0)$ elements: $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m$; we shall understand $\gamma_\beta + 1$ to be the length of the chain $\mathcal{C}_\beta$, and $e_\beta(t)$ to be the state existing at the (t + 1)th place from the origin in the chain $\mathcal{C}_\beta$ $(0 \leq t \leq \gamma_\beta)$.

4°. Let $P_0 = \| \tilde{p}_{ik} \|$ be a square matrix of order m obtained from the matrix $P = \| p_{ik} \|$ by the following rule:

$$\tilde{p}_{ik} = \begin{cases} p_{ik} & \text{if the transition } E_i \rightarrow E_k \text{ is contained in at least one minimal cycle:} \\ 0 & \text{in the oposite case} \end{cases} \tag{2.4}$$

The matrix P_0 is, generally speaking, decomposable. In its normal form all blocks are isolated since, according to (2.4) $\tilde{p}_{ik} > 0$ implies that $\tilde{p}_{ki}^{(N)} > 0$ at some natural value N; here $\| \tilde{p}_{ki}^{(N)} \| = P_0^N$. The decomposition of P_0 into blocks indexes the partitioning of the states of the chain into groups: the block formed by elements from the lines and columns with the numbers $i_1, i_2, \ldots, i_s$ of the matrix P_0 is placed in correspondence with the group of states $E_{i_1}, E_{i_2}, \ldots, E_{i_s}$. We shall employ $B_1, B_2, \ldots, B_\eta$ to denote non-zero blocks in the normal form of matrix P_0 and $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_\eta$ the groups of states corresponding to them. In the course of the proof of Lemma 5 of [5] one can be convinced of the validity of the following statement:

Lemma 1. The cycle $(E_{i_1}, E_{i_2}, \ldots, E_{i_l})$ is minimal if and only if these conditions are satisfied: $\tilde{p}_{i_k i_{k+1}} > 0$ ($k = 1, 2, \ldots l$; $i_{l+1} = i_1$).

Lemma 1 permits one to explain the theoretical stochastic sense of the partioning of the states of the chain into groups as defined above. Thus, we obtain:

a) All states of group $\mathcal{O}_h$ $(1 \leq h \leq \eta)$ belong to one general minimal cycle;

b) No two states of the different groups $\mathcal{O}_{h_1}$ and $\mathcal{O}_{h_2}$ belong to the same minimal cycle;

c) The zero blocks in the normal form of P_0 correspond to individual states which do not belong to any minimal cycle.

5°. For the chains $\mathcal{C}_\beta$ (refer to Point 3, Section 2) we shall define the characteristics $L_\beta(t)$, $H_\beta(t)$, and $K_\beta(t)$ $(0 \leq t \leq \gamma_\beta$, $1 \leq \beta \leq m)$ by the equalities

$$L_\beta(t) = l, \quad \text{if } e_\beta(t) = E_l ;$$

$$H_\beta(t) = \begin{cases} h & \text{if } e_\beta(t) \in \mathcal{O}_h ; \\ 0 & \text{if } e_\beta(t) \bar{\in} \mathcal{O}_h ; \end{cases} \qquad (h = 1, 2, \ldots, \eta) \tag{2.5}$$

$K_\beta(t) = k$ if for some $h (1 \leq h \leq \eta)$ the following are satisfied:

$e_\beta(k + \tau) \in \mathcal{O}_h$ $(\tau = 0, 1, \ldots t - k)$, $e_\beta(k - 1) \bar{\in} \mathcal{O}(h)$ (in case k = 0 the last condition drops out); $K_\beta(t)$ is not defined if $H_\beta(t) = 0$.

Let us agree that the chain $\mathcal{C}_\beta$ intersects the group of states $\mathcal{S}_h$ r $(r \geqq 1)$ times if the values of t $(0 \leqq t \leqq \gamma_\beta)$ include some such that $H_\beta(t) = h$, $K_\beta(t) = t$ precisely r different $(1 \leqq \beta \leqq m)$.

The method for finding all chains of length $n + 1$ for which $\mathcal{C}_\beta$ is the reduced chain is given by

<u>Lemma</u> 2. In order that the chain (2.2) have $\mathcal{C}_\beta$ as a reduced chain, it is necessary and sufficient that (2.2) can be obtained from $\mathcal{C}_\beta$ by adding, after state e (t) one of the minimal cycles ending with state $e_\beta(t)$ $(H_\beta(t) > 0,\ 0 \leq t \leq \gamma_\beta)$, observing the following condition at this time: the cycle added after e (t) must not contain the state $e_\beta(\tau)$ where $K_\beta(t) \leq \tau < t$ (for those t for which $K_\beta(t) = t$ this condition drops out).

The plan for proving Lemma 2 is clear from Points 1 and 2 of Section 2; carrying out the proof does not give rise to difficulties since it is possible to indicate in the reduced chain all those states after which the minimal cycles could be removed from the initial chain and this permits reconstruction of the complete original reduced chain in the set of all chains of length $n + 1$.

Generally speaking, it may turn out that for a given value of n the chain $\mathcal{C}_\beta$ is not a reduced chain for any chain of length $n + 1$. The problem of which values of n correspond to a fixed reduced chain is solved by

<u>Lemma</u> 3. Let the chain $\mathcal{C}_\beta$ intersect groups $\mathcal{S}_{h_1}, \ldots, \mathcal{S}_{h_\mu}$ and not intersect any other groups $\mathcal{S}_h$; let j_h be the index of imprimitivity of the matrix B_h; and let J_β be the greatest common divisor of the numbers $j_{h_1}, \ldots, j_{h_\mu}$. Then:

a) If chain $\mathcal{C}_\beta$ is a reduced chain for some chain of length $n + 1$, then

$$n \equiv \gamma_\beta \pmod{J_\beta} \tag{2.6}$$

b) For all sufficiently large values of n such that the condition (2.6) is satisfied, there exist chains of length $n + 1$ for which is a reduced chain.

<u>Proof</u>. In classifying irreducible Markov chains ([1], [3]) it is proved that the greatest common divisor (abbreviated to g. c. d.) of the lengths of all cycles passing through a fixed state is equal to the index of imprimitivity of the transition probability matrix. The application of this proposition to groups to states $\mathcal{S}_{h_k}$ $(1 \leq k \leq \mu)$ yields $\tilde{p}_{ii}^{(N)} = 0$ if $N \not\equiv 0 \pmod{j_h}$, $E_i = \mathcal{S}_h$, which, together with Lemma 2, permits one to conclude that in order for a chain of length $n + 1$ with a reduced chain $\mathcal{C}_\beta$ to exist, it is necessary to satisfy the condition (2.6).

We shall prove the second statement of Lemma 3 by the direct construction of chains of length $n + 1$ by the given reduced chain. Let $t = K_\beta(t)$; we shall choose minimal cycles ending at $e_\beta(t)$ with a greatest common divisor of lengths equal to j_h $(h = H_\beta(t)$ and substitute them in succession, one after another, in chain C_β after the state $e_\beta(t)$ $(0 \leqq t \leqq \gamma_\beta)$. As a result, the original composition of chain C_β is supplemented by cycles whose greatest common divisor of lengths is equal to J. We shall begin to change the length of the chains that are formed, repeating any of the added cycles an arbitrary number of times; the possibilities available in this direction are described by a well known lemma of number theory:

If $x_1, x_2, \ldots, x_N$ are natural numbers with the greatest common divisor d, then any sufficiently large natural number M such that $M = 0 \pmod d$ can be represented in the form $M = \sum_{i=1}^{N} a_i x_i$ where a_i are non-negative integers. Choosing the lengths of the added cycles for x_i and assuming that $M = h - \gamma_\beta$, we see that in order to carry out the planned construction it is sufficient to repeat the cycle of length x_i $(i = 1,2,\ldots)$ a_i times when expanding chain C_β.

We shall complete consideration of the structure of chains of length $n + 1$ and weight $\delta n + \gamma$ with this and pass on to study their stochastic properties.

3. The Local Limit Theorem for Extreme Values $\sum_{k=0}^{n} f(e(k))$

1°. We shall use $P_\beta^{(n)}(t)$ to denote the probability that the chain $\{e(0)e(1)\ldots e(n)\}$ $(e(0) = E_q)$ will have $\{e_\beta(0)e_\beta(1)\ldots e_\beta(t)\}$ as a reduced chain. Since the transformation described in Point 2 of Section 2 of chains into reduced chains is unique and applicable to any chain, the following formula is valid:

$$P_{qj}(n, \delta n + \gamma) = \sum_{\beta=1}^{m} P_\beta^{(n)}(\gamma_\beta). \tag{3.1}$$

Henceforth we shall study the behavior of each component $P_\beta^{(n)}(\gamma_\beta)$ separately.

2°. Let us study the chain $C_\beta (1 \leqq \beta \leqq m)$. The state $e_\beta(t)$ $(0 \leqq t \leqq \gamma_\beta)$ is such that $H_\beta(t) > 0$ corresponds to the transition probability matrix whose elements belong to those minimal cycles which can be substituted after $e_\beta(t)$ to expand the chain C_β in the sense indicated in Lemma 2. We shall write this matrix as $B_\beta(t)$. When $t = K_\beta(t)$, matrix $B_\beta(t)$ is equal to B_h $(h = H_\beta(t))$ and when $t > K_\beta(t)$, matrix $B_\beta(t)$ is obtained from B_h by striking outlines and columns corresponding to the states $e_\beta(\tau)$ $(\tau = K_\beta(t), K_\beta(t) + 1, \ldots, t - 1)$.

Let $\pi^{(n)}(t)$ be the element of matrix $B_\beta^{(n)}$ located at the intersection of the line and the column corresponding to the state $e_\beta(t)$. Following the well known reasoning of the theory of Markov chains, we conclude that $\pi^{(n)}(t)$ is equal to the probability that the sequence of states $e(0), e(1), \ldots, e(n-1)$ ($e(0) = e\,(t)$) constitutes a minimal cycle which does not contain $e_\beta(k), e_\beta(k+1), \ldots, e_\beta(t-1)$ ($k = k\,(t)$; if $K_\beta(t) = t$, then all permitted states are solved in the minimal cycle) and in this case it turns out that $e(n) = e_\beta(t)$. Since $B_\beta(t)$ is an irreducible matrix with non-negative elements, the asymptotic expression for $\pi_\beta{}^{(n)}$ is given by the Perron formula (refer, for example, to [3]):

$$\pi_\beta{}^{(n)}(t) = \begin{cases} j_\beta(t)\,\dfrac{A_\beta(t)}{C_\beta(t)}\,\lambda_\beta{}^n(t) + O((\lambda_\beta(t) - \varepsilon)^n) & \text{if } n \quad 0 \pmod{j_\beta(t)} \\ 0 & \text{if } n \quad 0 \pmod{j_\beta(t)}. \end{cases} \tag{3.2}$$

Here:

$\lambda_\beta(t)$ is the largest positive root of the equation

$$\det \| E - B_\beta(t) \| = 0 \tag{3.3}$$

$A_\beta(t)$ is the algebraic complement to the element $\lambda_\beta(t) - \pi_\beta{}^{(1)}(t)$ in the matrix $\lambda_\beta(t)E - B_\beta(t)$;

$$C_\beta(t) = \frac{\partial}{\partial\lambda}\left[\det \|\lambda E - B_\beta(t)\|\right]_{\lambda = \lambda_\beta(t)};$$

$j_\beta(t)$ is the index of imprimitivity of the matrix $B_\beta(t)$;
ε is some positive number not larger than $\lambda_\beta(t)$.

If $t > K(t)$, then according to Wielandt's lemma ([4], Chapter XIII) the inequality $\lambda_\beta(t) < \lambda_\beta(t-1)$ holds, consequently,

$$\pi_\beta{}^{(n)}(t) = O(\lambda_\beta(k) - \varepsilon_1)^n), \quad 0 \leqq \varepsilon_1 < \lambda_\beta(k) \text{ when } t > K_\beta(t) = k \tag{3.4}$$

We note also that Wielandt's lemma implies that $\lambda_\beta(t)$ is smaller than the maximum positive number of the matrix $P = \| p_{ik} \|$ if only not all cycles of the chain under consideration are minimal ($t = 1, 2, \ldots, \gamma_\beta$). When the specific weights of all cycles are the same, we have $h = 1$, $B_1 = P$.

3°. The method set forth below for singling out the principal term in $P_\beta{}^{(n)}(t)$ ($0 \leqq t \leqq \gamma_\beta$; $1 \leqq \beta \leqq m$) when $n \to \infty$ is connected with the formulas (3.2) and (3.4). First we shall introduce the following notation:

$$A_\beta(t) = \max_{1 \leqq k \leqq \mu} \lambda_{h_k}, \tag{3.5}$$

where λ_{h_k} is the largest of the absolute values of the characteristic numbers of matrix B_{h_k};

$$J_\beta(t) = \underset{1 \leqq k \leqq \mu}{\text{g. c. d.}} \{j_{h_k}\}, \tag{3.6}$$

where j_{h_k} is the index of primitivity of matrix B_{h_k};

$$I_\beta(t) = \text{g.c.d.} \ \{j_{h_k}\} \ \text{for all } h_k \text{ for which } \lambda_{h_k} = \Lambda_\beta(t), \tag{3.7}$$

$$R_\beta(t) = \sum r_{h_k} \ \text{for all } h_k \text{ for which } \lambda_{h_k} = \Lambda_\beta(t). \tag{3.8}$$

Theorem 1. Let us assume that the chain $\{e_\beta(0)e_\beta(1)\ldots e_\beta(t)\}$ intersects the group of states $\mathcal{O}_{h_k}$ ($k = 1, 2, \ldots, \mu$) J_k times and that it does not intersect other groups $\mathcal{O}_h$.

Let $n_i \to \infty$ ($i = 0, 1, 2, \ldots$) by the law: $n_i \equiv \nu + t$ (mod $I_\beta(t)$, $\nu \equiv 0$ (mod $I_\beta(t)$), (ν is a fixed number). Then there exists a finite positive limit

$$\lim_{i \to \infty} \frac{P_\beta^{(n)}(t)}{n_i^{R_\beta(t)-1} \Lambda_\beta^{n_i(t)}} = \Pi_\beta^{(\nu)}(t).$$

We shall prove this by induction. The statement of Theorem 1 for $t = 0$ follows from (3.2). When $t > 0$, we have:

$$P_\beta^{(n)}(t) = \begin{cases} P_\beta^{(n-1)}(t-1) p_{\tau_1 \tau_2} \ \text{if } H_\beta(t) = 0; \text{ here } \tau_1 = L_\beta(t-1); \\ \qquad \tau_2 = L_\beta(t); \\ \sum_{k=1}^{n} P_\beta^{(k-1)}(t-1) p_{\tau_1 \tau_2} \pi_\beta^{(n-k)}(t) \ \text{if } H_\beta(t) > 0 \end{cases}$$

(refer to Point 2 of Section 3).

(3.9)

Let us assume that Theorem 1 is valid for some value $t - 1$; then its validity for t in case H $(t) = 0$ is certain. Let $H_\beta(t) > 0$, then we shall divide the possibilities represented here into three cases:

a) $\lambda_\beta(t) = \Lambda_\beta(t-1)$, b) $\lambda_\beta(t) > \Lambda_\beta(t-1)$, c) $\lambda_\beta(t) < \Lambda_\beta(t-1)$

a) In this case: $R_\beta(t) = R_\beta(t-1) + 1$. Formula (3.9) leads to the relationship

$$\frac{P_\beta^{(n)}(t)}{n^{R_\beta(t)-1}\Lambda_\beta^{(n)}(t)} = \frac{P_{\tau_1\tau_2}}{n\Lambda_\beta(t)} \sum_{k=1}^{n} \frac{P_\beta^{(k-1)}(t-1)}{(k-1)^{R_\beta(t-1)}\Lambda_\beta^{k-1}(t-1)}$$

$$\left(\frac{k-1}{n}\right)^{R_\beta(t)-2} \frac{\Pi_\beta^{(n-k)}(t)}{\Lambda_\beta^{n-k}(t)} . \qquad (3.10)$$

On the strength of the previously cited lemma from number theory (refer to the proof of Lemma 3), for each sufficiently large natural value ν which satisfies the condition $\nu \equiv 0 \pmod{J_\beta(t)}$, there exist non-negative integers $Y_1(\nu)$, $Y_2(\nu)$ such that

$$Y_1(\nu) + Y_2(\nu) = \nu ; \quad Y_1(\nu) = 0 \pmod{J_\beta(t-1)};$$
$$Y_2(\nu) = 0 \pmod{j_\beta(t)}.$$

We shall use $y_i(\nu)$ to denote the smallest of the possible values of $Y_i(\nu)$ $(i = 1, 2)$ with a fixed ν, and $\varphi_\beta(t)$ to denote the least common multiple (abbreviated l.c.m.) of $J_\beta(t-1)$ and $j_\beta(t)$. With a sufficiently large value of ν, the numbers $y_1(\nu) + k\,\varphi_\beta(t)$

$$k = 0, 1, \ldots \frac{\nu - y_1(\nu)}{\varphi_\beta(t)} \quad \text{include} \quad \psi_\beta(t) = \frac{I_\beta(t-1)}{\text{l.c.m.}\{I_\beta(t-1), \varphi_\beta(t)\}}$$

of different residues modulo $I_\beta(t-1)$; let $w_1(\nu)$ be the smallest non-negative one of these numbers. We shall establish the validity of the following statement:

<u>Lemma 4</u>. If there exists a d such that for any sufficiently large natural n

$$w_1(n + d) = w_1(n), \qquad (3.11)$$

then the sequence $\dfrac{P_\beta^{(N_i)}(t)}{N_i^{R_\beta(t)-1}\Lambda_\beta^{N_i}(t)}$ $(N_i = \nu + t + id; \ i = 0, 1, \ldots)$ has a limit.

Indeed, setting $n = \nu + t + id$ and rejecting terms known to be equal to zero, we obtain

$$\frac{P_\beta^{(N_i)}(t)}{N_i^{R_\beta(t)-1}\Lambda_\beta^{N_i(t)}} = \frac{P_{l_1 l_2}}{\Lambda_\beta(t)N_i} \sum_{k=0}^{\frac{N_i-y_1(\nu)-t}{\varphi_\beta(t)}} \frac{P_\beta^{(\vartheta_k)}(t-1)}{\vartheta_k^{R_\beta(t-1)-1}\Lambda_\beta^{\vartheta_k(t-1)}} \times$$

$$\times \left(\frac{\vartheta_k}{N_i}\right)^{R_\beta(t)-2} \frac{\pi_\beta^{(N_i-1-\vartheta_k)}(t)}{\Lambda_\beta^{N_i-1-\vartheta_k}(t)} \qquad (3.12)$$

where $\vartheta_k = y_1(\nu) + t - 1 + k\varphi_\beta(t)$. It can be seen from (3.12) that the expression for $\dfrac{P_\beta^{(N_i)}(t)}{N_i^{R_\beta(t)-2}\Lambda_\beta^{N_i}(t)}$ can differ, with fixed i from the sum

$$\sum_i = \frac{P_{l_1 l_2}}{\Lambda_\beta(t)} \sum_{k=0}^{\varphi_\beta(t)-1M_i} \sum_{l=0} \frac{P_\beta^{(\vartheta_{kl})}(t-1)}{k^{R_\beta(t-1)-1}\Lambda_\beta^{\vartheta_{kl}}(t-1)} \times \frac{\vartheta_{kl}}{N_i}^{R_\beta(t)-2}$$

$$\frac{\pi_\beta^{(N_i-\vartheta_{kl}-1)}(t)}{\Lambda_\beta^{N_i-\vartheta_{kl}-1}(t)},$$

where

$$\vartheta_k = w_1(\nu) + t - 1 + k\varphi_\beta(t) + l\chi_\beta(t); \quad \chi_\beta(t) = \text{l.c.m.}\left\{I_\beta(t-1), \varphi_\beta(t)\right\} = \text{l.c.m.}\left\{J_\beta(t-1), j_\beta(t)\right\},$$

$$M_i = \frac{N_i - w_i(\nu) - t - k\varphi_\beta(t)}{\chi_\beta(t)},$$

only the absence of certain terms whose total number does not exceed a constant independent of i. Consequently,

$$\lim_{i\to\infty} \frac{P_\beta^{(N_i)}(t)}{N_i^{R_\beta(t)-1}\Lambda_\beta^{N_i}(t)} - \frac{\sum_i}{N_i} = 0. \qquad (3.13)$$

Bearing in mind the assumption of the induction, formula (3.2), and the equality

$$\lim_{i \to \infty} \frac{\Sigma_i}{N_i} = \frac{p\,\ell_1 \ell_2}{\Lambda_\beta(t)} \cdot \frac{j_\beta(t)}{(R_\beta(t) - 1)\chi_\beta(t)} \cdot \frac{A_\beta(t)}{U_\beta(t)} \sum_{k=0}^{\psi_\beta(t)-1} \Pi_\beta^{(w_1(\nu)+k\varphi_\beta(t))}(t-1)$$

This result and the relationship (3.13) include the statement of Lemma 4.

We shall complete consideration of case a) by proving that the smallest of the natural values of d with which (3.11) is satisfied independently of n is the number I(t). Let us turn to the original equalities:

$$y_1(\nu) \equiv 0 \pmod{J_\beta(t-1)}; \quad y_1(\nu) \equiv \nu \pmod{j_\beta(t)}$$

$$y_1(\nu+d) \equiv 0 \pmod{J_\beta(t-1)}; \quad y_1(\nu+d) \equiv \nu + d \pmod{j_\beta(t)}. \qquad (3.14)$$

According to the definition of $w_1(\nu)$, the equality (3.11) is possible if and only if there exists an integer such that

$$y_1(\nu+d) - y_1(\nu) \equiv \chi \cdot \varphi_\beta(t) \pmod{I_\beta(t-1)}. \qquad (3.15)$$

As is well known, in order that the congruence (3.15) be solvable for x, it is necessary and sufficient that the following condition should be satisfied:

$$y_1(\nu + d) - y_1(\nu) = 0 \pmod{\text{g.c.d.}\{I_\beta(t-1),\ \varphi_\beta(t)\}},$$

$$\text{g.c.d.}\ \{I_\beta(t-1),\ \varphi_\beta(t)\} = J_\beta(t-1)\ \frac{\text{g.c.d.}\ \{j_\beta(t),\ I_\beta(t-1)\}}{\text{g.c.d.}\ \{j_\beta(t),\ I_\beta(t-1),\ I_\beta(t)\}} =$$
$$= \text{l.c.m.}\ \{J_\beta(t-1),\ I_\beta(t)\} .$$

The difference $y_1(\nu + d) - y_1(\nu)$ is divided by the l.c.m. $\{J_\beta(t-1), I_\beta(t)$ if and only if it is divided by each of the numbers $J_\beta(t-1)$ separately. Carrying out paired subtraction of congruences in (3.14), we obtain:

$$y_1(\nu + d) - y_1(\nu) = \begin{matrix} 0 \bmod J_\beta(t-1) \\ d \bmod I_\beta(t). \end{matrix}$$

Consequently, in order to satisfy the equality (3.11), it is necessary and sufficient that d be divided by $I_\beta(t)$, which was required to be proven.

b) Let $\lambda_\beta(t) > \Lambda_\beta(t-1)$; then $\Lambda_\beta(t) = \lambda_\beta(t)$, $R_\beta(t) = 1$, and $I_\beta(t) = j_\beta(t)$. Setting n equal to $\nu + t + iI_\beta(t)$ $(=n_i)$ in (3.9) and ordering the writing of the terms by the same method, as in case a), we arrive at a relationship analogous to (3.12):

$$\frac{P_\beta^{(n_i)}(t)}{\Lambda_\beta^{n_i}(t)} = \frac{P_{\ell_1\ell_2}}{\Lambda_\beta(t)} \sum_{\ell=0}^{\frac{n_i - y_1(\nu) - t}{\varphi_\beta(t)}} \frac{P_\beta^{(\vartheta_k)}(t-1)}{\vartheta_k^{R_\beta(t-1)-1}\Lambda_\beta^{\vartheta_k}(t-1)} \times$$

$$\times \vartheta_k^{R_\beta(t-1)-1} \left(\frac{\Lambda_\beta(t-1)}{\Lambda_\beta(t)}\right)^{\vartheta_k} \times \frac{\Pi_\beta^{(n_i-1-\vartheta_k)}(t)}{\Lambda_\beta^{n_i-1-\vartheta_k}(t)}.$$

Here, as distinguished from the case considered previously, the quantities $n_i - 1 - \vartheta_0$ $(i = 0, 1, 2, \ldots)$ must have the same residue modulo $j_\beta(t)$ and this permits us to write for $\frac{P_\beta^{(n_i)}(t)}{\Lambda_\beta^{n_i}(t)}$ an exact formula in the form:

$$\frac{P_\beta^{(n_i)}(t)}{\Lambda_\beta^{n_i}(t)} = \frac{P_{\ell_1\ell_2}}{\Lambda_\beta(t)} \cdot \sum_{k=0}^{\psi_\beta(t)-1} \sum_{\ell=0}^{\tilde{M}_i} \frac{P_\beta^{(\vartheta_{k\ell})}(t-1)}{\tilde{\vartheta}_k^{R_\beta(t-1)-1}\Lambda_\beta^{\tilde{\vartheta}_{k\ell}}(t-1)} \times$$

$$\times \tilde{\vartheta}_k^{R_\beta(t-1)-1} \cdot \left(\frac{\Lambda_\beta(t-1)}{\Lambda_\beta(t)}\right)^{\tilde{\vartheta}_k} \cdot \frac{\Pi_\beta^{(n_i-1-\tilde{\vartheta}_k)}(t)}{\Lambda_\beta^{n_i-1-\tilde{\vartheta}_k}(t)},$$

where

$$\tilde{\vartheta}_{k\ell} = y_1(\nu) + t - 1 + k\varphi_\beta(t) + \ell\chi_\beta(t); \quad \tilde{M}_i = \frac{n_i - y_1(\nu) - t - k\varphi_\beta(t)}{\chi_\beta(t)}.$$

Since on the assumption of induction the quantity $\frac{P_\beta^{(\tilde{\vartheta}_{k\ell})}(t-1)}{\tilde{\vartheta}_{k\ell}^{R_\beta(t-1)-1}\Lambda_\beta^k(t-1)}$ tends to $\Pi_\beta(y_1(\nu)+k\varphi_\beta(t))(t-1)$ when $\ell \to \infty$ and since, according to (3.2) and (3.4), the series $\sum_{\ell=0}^{\infty} \tilde{\vartheta}_{k\ell}^{R_\beta(t-1)-1} \times$ $\times \left(\frac{\Lambda_\beta(t-1)}{\Lambda_\beta(t)}\right)^{\tilde{\vartheta}_{k\ell}}$ converges absolutely to a certain sum $\sigma_\beta^{(k)}(t)$

while

$$\frac{\Pi_\beta(n_i-1-\tilde{\vartheta}_{k\ell})}{\Lambda_\beta^{n_i-1-\tilde{\vartheta}_{k\ell}}} \to j_\beta(t)\frac{A_\beta(t)}{C_\beta(t)} \text{ when } i \to \infty ,$$

then the limit of the right hand side of (3.16) when $i \to \infty$ exists and is equal to

$$\frac{p_{\ell_1 \ell_2}}{\Lambda_\beta(t)} j_\beta(t) \frac{A_\beta(t)}{C_\beta(t)} \sum_{k=0}^{\psi_\beta(t)-1} \sigma_\beta(k)(t)\, \Pi_\beta(y_1(\nu)+k\varphi_\beta(t))_{(t-1)}.$$

c) In case $\lambda_\beta(t) < \Lambda_\beta(t-1)$ we have:

$$\Lambda_\beta(t) = \Lambda_\beta(t-1); \quad R_\beta(t) = R_\beta(t-1); \quad I_\beta(t) = I_\beta(t-1).$$

Let us substitute the quantity $n_i = \nu + t + iI_\beta(t)$ in place of n and substitute the expression obtained to the form:

$$\frac{P_\beta^{(n_i)}(t)}{n_i^{R_\beta(t)-1}\Lambda_\beta^{n_i}(t)} = \frac{p_{\ell_1\ell_2}}{\Lambda_\beta(t)} \sum_{k=0}^{\psi_\beta(t)-1} \sum_i \frac{n_i-1-y_2\nu-k\varphi_\beta(t)}{\chi_\beta(t)}$$

where

$$\sum_i(n) = \sum_{\ell=0}^{n} \frac{\Pi_\beta(\chi_{k\ell})(t)}{\Lambda_\beta^{\chi_{k\ell}}(t)} \cdot \frac{P_\beta^{(n_i-1-\chi_{k\ell})}(t-1)}{(n_i-1-\chi_{k\ell})^{R_\beta(t-1)-1}\Lambda_\beta^{n_i-1-\chi_{k\ell}}(t-1)} \times$$

$$\times \left(\frac{n_i-1-\chi_{k\ell}}{n_i}\right)^{R(t)-1}$$

$$\chi_{k\ell} = y_2(\nu) + k\varphi_\beta(t) + \ell\chi_\beta(t)$$

(refer to case a) in regard to other notation). Based on the assumption of induction and the use of formula (3.4), we derive the evaluation

$$\left|\sum_i(n) - \sum_i(n')\right| < \varepsilon \text{ for all sufficiently large } i, n, n';$$

$n_i \geq n \geq n_0(\varepsilon)$ with an arbitrary positive ε. This implies that when

, the ratio $\dfrac{P_\beta{}^{(n_1)}(t)}{n_1{}^{R_\beta(t)-1}\Lambda_\beta{}^{n_1}(t)}$ tends to a finite limit

written as

$$\frac{P_{\zeta_1\zeta_2}}{\Lambda_\beta(t)}\sum_{k=0}^{\psi_\beta(t)-}\tau_\beta(k)(t)\,\Pi_\beta(\nu-y_2(\nu)-k\,\varphi_\beta(t))(t-1)$$

where

$$\tau_\beta(k)(t)=\sum_{\zeta=0}^{\infty}\frac{\Pi_\beta(y_2(\nu)+k\varphi_\beta(t)+\zeta\chi_\beta(t))(t)}{\Lambda_\beta y_2(\nu)+k\varphi_\beta(t)+\zeta\chi_\beta(t)(t)}\;.$$

4°. Substitution of the values $P_\beta{}^{(n)}(\gamma_\beta)$ given by Theorem 1 when $t=\gamma_\beta$ into (3.1) leads to the local limit theorem for extreme values of $\sum_{k=0}^{n} f(e(k))$. The corresponding formulation was given in Section 1; the sense of the notation used at this time is the same as in Section 3 (refer to Point 1° of Section 3, (3.5), (3.6), (3.7), (3.8), and also Point 3° of Section 2). There is no need for special derivation of the integral limit theorem for extreme values of $\sum_{k=0}^{n} f(e(k))$ since the probability $P\left\{\sum_{k=0}^{n} f(e(k))\leqq\delta n+\Gamma,\ e(n)=E_j\,|\,E(0)=E_q\right\}$ by summing the quantities $P_{qj}(n,\ \delta n+\nu)$ over ν to a number not exceeding some of the constants which do not depend on n.

5°. Recurrence relationships originating in the results of Point 3 of Section 3 can be used for actual determination of the values of $\Pi_\beta(\nu)(t)$:

$$\Pi_\beta(\nu)(t)=\frac{P_{\zeta_1\zeta_2}}{\Lambda_\beta(t)}\cdot j_\beta(t)\cdot\frac{A_\beta(t)}{C_\beta(t)}\cdot\frac{1}{(R_\beta(t)-1)\chi_\beta(t)}\sum_{k=0}^{\psi_\beta(t)-1}\Pi_\beta(w_1(\nu)+k\,\varphi_\beta(t))(t-1),$$

if $\lambda_\beta(t)=\Lambda_\beta(t-1)$

$$\pi_\beta^{(\nu)}(t) = \frac{p_{\zeta_1\zeta_2}}{\Lambda_\beta(t)} \cdot j_\beta(t) \cdot \frac{A_\beta(t)}{C_\beta(t)} \cdot \sum_{k=0}^{\psi\beta(t)-1} \sigma_\beta(k)(t)\, \pi_\beta(y_1(\nu)+k\varphi_\beta(t))(t-1),$$

if $\Lambda_\beta(t) > \Lambda_\beta(t-1)$

$$\pi_\beta(\nu)(t) = \frac{p_{\zeta_1\zeta_2}}{\Lambda_\beta(t)} \sum_{k=0}^{\psi\beta(t)-1} \tau_\beta(k)(t) \cdot \pi_\beta(\nu - y_2(\nu) - k\varphi_\beta(t))(t-1),$$

if $\lambda_\beta(t) < \Lambda_\beta(t-1)$ with the initial condition: $\pi_\beta{}^{(0)}(\tau) = j_\beta(\tau)\, \frac{A_\beta(\tau)}{C_\beta(\tau)}$

where τ is the least of the values of $t(0 \leq t \leq \gamma_\beta)$ for which $H_\beta(t) > 0$.

Another method for finding the limits of $\pi_\beta^{(\nu)}(t)$ will be shown in the next section.

4. The Exact and Asymptotic Formulas for $P_{qj}(n, \delta n + \gamma)$

1°. As is well known (refer, for example, to [3], Chapter IV), the explicity expression for the generating function $\sum_s P_{qj}(n, s) . z^s$ for fixed z is given by the formula

$$\sum_s P_{qj}(n, s)\, z^s = z^{s_q} \sum_{k=1}^{m} \frac{1}{(m_k(z))!} \cdot$$

$$\frac{d^{m_k(z)-1}}{d\lambda^{m_k(z)-1}} \left[\frac{\lambda^{n}(z) . D_{qj}(z, \lambda)}{D_k(z, \lambda)} \right]_{\lambda = \lambda_k(z)} \qquad (4.1)$$

Here:

$\lambda_1(z), \lambda_2(z), \ldots, \lambda_m(z)$ are characteristic numbers of the matrix $\lambda E - \| p_{ik} z^{s_k} \|$; $m_k(z)$ is the number of $\lambda_i(z) (1 \leq i \leq m)$ equal to $\lambda_k(z)$; $D_{qj}(z, \lambda)$ is the algebraic complement to the element $\lambda\delta_{jq} - p_{jq} z^{s_q}$ in the matrix $\lambda E - \| p_{ik} z^{s_k} \|$.

$$D_k(z, \lambda) = \frac{\det(\lambda E - \| p_{ik} z^{s_k} \|)}{(\lambda - \lambda_k(z))^{m_k(z)}} .$$

In a sufficiently small neighborhood of zero from which the points of one of the radii have been removed one can consider that

a) $\lambda_1(z), \lambda_2(z), \ldots, \lambda_m(z)$ as functions of z are unique analytic branches of the algebraic function $\lambda(z)$ implicitly given by the equation $\det(\lambda E - \| p_{ik} z^{s_k} \| = 0$;

b) $m_k(z)$ does not depend on z $(k = 1, 2, \ldots, m)$;

c) The function $\left. \frac{d^i}{d\lambda^i} \frac{D_{qj}(z, \lambda)}{D_k(z, \lambda)} \right|_{\lambda = \lambda_k(z)}$ is expanded into a series in (fractional) powers of z, of which only a finite number are negative $(i \leqq mk(z) - 1, \ k = 1, 2, \ldots, m)$.

Further, on the strength of Lemma 5, the functions $\lambda_k(z)$ $(k = 1, 2, \ldots, m)$ have zeroes of order not lower than δ at the point $z = 0$.

Consequently, both the exact and the asymptotic formulas for $P_{qj}(n, \delta n + \gamma)$ can be obtained as a result of carrying out arithmetical operations on the coefficients of the power series in (4.1) in a number not exceeding some finite quantity which depends on γ but does not depend on n.

Let us assume that the chains forming the set $m_{qj}(\gamma)$ are known. In this case the application of the results of Section 3 to (4.1) gives, in general form, the explicit expression of the coefficient in the principal term in $P_{qj}(n, \delta n + \gamma)$. Moreover, as will be seen from what is to follow, the assumption which has been made will permit simplifying the calculation of $P_{qj}(n, \delta n + \gamma)$ with concrete values of p_{ik}, s_k $(1 \leqq i, k \leqq m)$.

2°. Generalizing the definition of Point 1 of Section 3, we shall introduce for consideration the probability $p_\beta{}^{(n)}(t, l)$ that the chain $\{e(0), e(1), \ldots, e(n)\}$ $(e(0) = E_q)$ will satisfy the conditions:

a) $e(n) = E_l$;

b) For some k $(0 \leqq k \leqq n)$, the following are satisfied:

(1) $\{e_\beta(0) e_\beta(1) \ldots e_\beta(t)\}$ serves as the reduced chain for the chain $\{e(0), e(1), \ldots, e(k)\}$;

(2) $e(k + \tau) \in G_h$ $(1 \leqq \tau \leqq n - k)$, here $h = H_\beta t)$;

(3) $e(k+\tau) \neq e_\beta(t')$ $(1 \leqq \tau \leqq n-k,\ K_\beta(t) \leqq t' < t)$.

The explicit expression $P_\beta^{(n)}(t, i)$ yields

Theorem 2. The following equality is valid for all natural values of n:

$$P_\beta^{(n)}(t, i) = p_\beta(t) \cdot \sum_{k=1}^{M} \frac{1}{(m_k-1)!} \cdot \frac{d^{m_k-1}}{d\eta^{m_k-1}}$$

$$\left[\eta^{n-T_0} \cdot (\eta-\eta_k)^{m_k} \left(\prod_{l=1}^{T-1} \frac{A_\beta(\sigma_l, \eta)}{G_\beta(\tau_l, \eta)} \cdot \frac{A_\beta^{(i)}(\sigma_T, \eta)}{G_\beta(\sigma_T, \eta)} \right)\right]_{\eta=\eta_k} \tag{4.2}$$

Here:

$\tau_1, \tau_2, \ldots, \tau_T$ is the subset of all different numbers which satisfy the conditions

$K_\beta(\tau) = \tau$, $0 \leqq \tau \leqq t\ \gamma_\beta$ $(t, \beta$ are fixed$)$; (4.3)

σ_l is the least of all numbers for which the following are satisfied:

$\tau \leqq \sigma \leqq \gamma_\beta$, $H_\beta(\sigma) = H_\beta(\tau_l)$, $H_\beta(\sigma+1) \neq H_\beta(\tau_l)$ $(1 \leqq l \leqq T)$; (4.4)

$\eta_1, \eta_2, \ldots, \eta_M$ are all different roots of the equation

$\prod_{l=1}^{T} \det(\eta E - B_\beta(\tau_l)) = 0$; m_k is the multiplicity of the root

η_k $(1 \leqq k \leqq M)$;

$$G_\beta(\tau, \eta) = \det(\eta E - B_\beta(\ )),$$

$A_\beta^{(l_1, l_2)}(\tau, \eta)$ is the algebraic complement to the element $\eta\delta_{l_2 l_1} - p_{l_2 l_1}$ in the matrix $\eta E - B_\beta(\tau)$

$$A_\beta^{(l_2)}(\tau, \eta) = A_\beta^{(l_1, l_2)}(\tau, \eta) \text{ when } l_1 = L_\beta(t);$$

$$A_\beta(\tau, \eta) = A_\beta^{(l)}(\tau, \eta) \text{ when } l = L_\beta(t) \tag{4.6}$$

$$p_\beta(t) = \prod_{\tau=0}^{t-1} p_{l_\tau l_\tau+1} \text{ where } l_\tau = L_\beta(\tau); \tag{4.7}$$

T is the number of different values of τ such that $K_\beta(\tau) = 0$, $0 \leqq \tau \leqq t \leqq \gamma_\beta$.

<u>Proof.</u> According to Lemma 2, the quantities $P_\beta^{(n)}(t)$, $P_\beta^{(n)}(t,i)$ (β, n are fixed; t, i run through all the values permitted in the sense of the definitions of Point 1 of Section 3 and Point 2 of Section 4 should satisfy the system of finite difference equations:

$$p_\beta^{(n)}(t) = p_\beta^{(n-1)}(t-1)\, p_{l_1 l_2}, \text{ if } H_\beta(t)=0,\ t \geq 1,\ n \geq 1, \text{ here } l_1 = L\,(t-1);$$

$$l_2 = L_\beta(t),$$

$$p_\beta^{(n)}(0) = \begin{cases} 0, & \text{if } H_\beta(0) \text{ and } n > 0, \\ \sum_i p_\beta^{(n-1)}(0,i) p_{il}, & \text{if } H_\beta(0) > 0 \text{ and } n > 0, \text{ here } l = L_\beta(0) \end{cases}$$

$$p_\beta^{(n)}(t,k) = \sum_i p_\beta^{(n-1)}(t,i) p_{ik}, \qquad \text{if } K_\beta(t) > t \tag{4.8}$$

$$p_\beta^{(n)}(t,k) = p_\beta^{(n)}(t), \qquad \text{if } k = K_\beta(t)$$

$$p_\beta^{(n)}(t) = p_\beta^{(n-1)} \cdot p_{l_1 l_2} + \sum_i p_\beta^{(n-1)}(t,i) p_{i l_2}, \text{ if } H_\beta(t) > 0,\ n \geq 1,\ t \geq 1.$$

$$\text{here } l_1 = L_\beta(t-1);\ l_2 = L_\beta(t),$$

with initial conditions:

$$P_\beta^{(0)}(0) = 1; \quad P_\beta^{(0)}(t) = 0 \text{ if } t > 0; \quad P_\beta^{(0)}(t,i) = 0,$$
$$\text{if } t^2 + [K\,(t) - i]^2 > 0.$$

Making use of the expression for solutions of the system (4.8) given by the general theory (refer, for example, to [3], Chapter 1), after transforming the determinants in these expressions with the aid of the Laplace theorem, we obtain:

$$P_\beta^{(n)}(t,i) = p_\beta(t) \cdot \sum_{k=1}^{M'} \frac{1}{(m'_k-1)!} \cdot \frac{d^{m'_k-1}}{d\eta^{m'_k-1}} \Bigg[\eta^{n-T} Q \cdot (\eta - \eta'_k)^{m'_k} \left(\prod_{\substack{H_\beta(\tau) > 0 \\ 0 \leq \tau < \sigma T}} \frac{A_\beta(\tau,\eta)}{C_\beta(\tau,\eta)} \times \frac{A_\beta^{(i)}(\sigma_T,\eta)}{C_\beta(\sigma_T,\eta)} \right) \Bigg]_{\eta = \eta'_k} \tag{4.9}$$

where η_k' is a root of multiplicity m_k' of the equation

$$\prod_{\substack{H_\beta(\tau)>0 \\ 0\leqq\tau\leqq\sigma_T}} C_\beta(\tau,\eta) = 0 \quad (1 \leqq k \leqq M').$$

Let m_k'' denote the number of factors $\eta - \eta_k$ in $\prod_{\substack{H_\beta(\tau)>0 \\ 0\leqq\tau\leqq\tau_T}} C_\beta(\tau,\eta)$.

Reducing the fractions in the right hand side of (4.9) and replacing the term corresponding to the quantity $\eta = \eta_k'$ with the expression equal to it

$$p_\beta(t)\, \frac{1}{(m_k''-1)!} \cdot \frac{d^{m_k''-1}}{d\eta^{m_k''-1}} \Bigg[\eta^{n-T} 0 (\eta - \eta'_k)^{m''_k} \cdot \left(\prod_{l=1}^{T-1} \frac{A_\beta(\sigma_l,\eta)}{C_\beta(\tau_l,\eta)} \right) \cdot \frac{A_\beta^{(i)}(\sigma_T,\eta)}{C_\beta(\sigma_T,\eta)} \Bigg]_{\eta=\eta'_k}$$

we arrive at formula (4.2), which proves the theorem.

3° Let us consider the equality

$$P_\beta^{(n)}(t,l) = P_\beta^{(n)}(\gamma_\beta) \text{ when } t = \gamma_\beta \text{ and } l = L_\beta(\gamma_\beta). \qquad (4.10)$$

Combined with (3.1) and (4.10), Theorem 2 gives an explicit expression for the probability $P_{qj}(n, \sigma n + r)$ which for all natural values n is the sum of those terms of the form $a_k n^{R_k} \lambda_k^n$ like the expression derived in Point 1° of Section 4, but differing from the latter by the form in which the coefficients a_k are written.

4°. Theorem 2 also permits finding explicit expressions for $\Pi_\beta^{(\nu)}(t)$ (refer to Points 3°, 5° of Section 3); we shall show this. Let the notation γ_β, $L_\beta(t)$, $\lambda_\beta(t)$, $\Lambda_\beta(t)$, $J_\beta(t)$, $I_\beta(t)$, $R_\beta(t)$, τ_l, σ_l, $C_\beta(\tau,\eta)$, $A_\beta(\sigma,\eta)$, and $p_\beta(t)$ be employed to maintain the sense indicated by (2.5), (3.3), (3.5), (3.6), (.7), (3.8), (4.3), (4.4), (4.5), (4.6), and (4.7) respectively in Point 3° of Section 2. From Perron's theorem and Wielandt's lemma ([4], Chapter XIII) we deduce:

a) The equation $\prod_{\tau_l\leqq t} C_\beta(\tau_l,\eta) = 0$ has the roots

$\Lambda_\beta(t) e^{2\pi i \frac{k}{I_\beta(t)}}$ $(k = 0, 1, \ldots, I_\beta(t) - 1)$ of multiplicity $R_\beta(t)$;

b) The multiplicity of the roots of the equation $\prod_{\tau_l\leqq t} C_\beta(\tau_l,\eta) = 0$

is equal in absolute value to $\Lambda_\beta(t)$, but are different from

$\Lambda_\beta(t)e^{2\pi i \frac{k}{I_\beta(t)}}$ $(k = 0, 1, \dots, I_\beta(t) - 1)$ less than $R_\beta(t)$;

e) If $\lambda_\beta(\tau_l) = \Lambda_\beta(t)$, then

$$\left.\frac{A_\beta(\sigma_l, \lambda)}{\frac{\partial}{\partial\lambda} C_\beta(\tau_l, \lambda)}\right|_{\lambda = \Lambda_\beta(t) e^{2\pi i \frac{k}{I_\beta(t)}}} =$$

$$= e^{2\pi i \frac{k(\tau_l - \sigma_l)}{I_\beta(t)}} \left.\frac{A_\beta(\sigma_l, \lambda)}{\frac{\partial}{\partial\lambda} C_\beta(\tau_l, \lambda)}\right|_{\lambda = \Lambda_\beta(t)}$$

$$(k = 0, 1, \dots, I_\beta(t) - 1).$$

Using these results as a basis, we shall single out in the right hand side of (4.2), where we have set $i = L_\beta(t)$, the coefficient of $n^{R_\beta(t)-1} \Lambda_\beta^n(t)$ for $n \equiv \nu + t \pmod{I_\beta(t)}$ and equate it, in accordance with Theorem 1 with the quantity $\Pi_\beta^{(\nu)}(t)$.

$$\Pi_\beta^{(\nu)}(t) = \frac{1}{(R_\beta(t) - 1)!} \frac{p_\beta(t)}{\Lambda^{t+\nu}_\beta(t)} \prod_{\substack{\lambda_\beta(\tau_l) = \Lambda_\beta(t) \\ \tau_l \le t}} \left.\frac{\lambda^{\sigma_l - \tau_l} A_\beta(\sigma_l, \lambda)}{\frac{\partial}{\partial\lambda} C_\beta(\tau_l, \lambda)}\right|_{\lambda = \Lambda_\beta(t)} \times$$

$$\times \sum_{k=0}^{I_\beta(t)-1} \prod_{\substack{\lambda\nu\lambda_\beta(\tau_l) < \Lambda_\beta(t) \\ \tau_l \le t}} \left.\frac{\lambda^{\sigma_l - \tau_l + 1} A_\beta(\sigma_l, \lambda)}{C_\beta(\tau_l, \lambda)}\right|_{\lambda = \Lambda_\beta(t) \cdot e^{2\pi i \frac{k}{I_\beta(t)}}}.$$

In conclusion we note that no restrictions were placed on the matrix $P = \| p_{ik} \|$ in this article other than the non-negativity of the elements p_{ik} $(1 \le i, k \le m)$.

Submitted to the Editorial Staff
13 May 1959.

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[Translator's Note: An English language summary of this article is given at the end of the article, on page 352 of the original.]

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